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STRONGLY LECH-INDEPENDENT IDEALS AND LECH'S CONJECTURE

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Abstract. We introduce the notion of strongly Lech-independent ideals as a generalization of Lech-independent ideals defined by Lech and Hanes, and use this notion to derive inequalities on multiplicities of ideals. In particular we prove that if $(R,\mathfrak{m})\to(S,\mathfrak{n})$ is a flat local extension of local rings such that S is the localization of a standard graded ring over a field at the homogeneous maximal ideal, $\mathfrak{m}S$ is the localization of a homogeneous ideal and is $\mathfrak{n}$-primary, then $e(R)\leq e(S)$.

1. introduction

Around 1960, Lech made the following remarkable conjecture on the Hilbert-Samuel multiplicities in [9]:

Conjecture 1.1. *Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local extension of local rings. Then $e(R)\leq e(S)$.*

As the Hilbert-Samuel multiplicity measures the singularity of a ring, this conjecture roughly means that the singularity of R is no worse than that of S if $(R,\mathfrak{m})\to(S,\mathfrak{n})$ is a flat local extension. This conjecture has now stood for sixty years and remains open in most cases. It has been proved in the following cases:

(1) $\dim R\leq 2$ [9];
(2) $S/\mathfrak{m}S$ is a complete intersection [9];
(3) R is a strict complete intersection [1];
(4) $\dim R=3$ and R has equal characteristic [12];
(5) R is a standard graded ring over a perfect field (localized at the homogeneous maximal ideal) [13].

For other results see [3], [4], [5] and [11]. In this paper the key concept is a new notion called *strongly Lech-independence*, which is a natural generalization of Lech-independence introduced in [10] and explored in [4]. By definition, an ideal $I\subset S$ is strongly Lech-independent if for any i, I^i/I^{i+1} is free over S/I, and a set of elements is strongly Lech-independent if it is a minimal generating set of a strongly Lech-independent ideal. Under strongly Lech-independence assumption, we can calculate the colength of powers of ideals using the data on the monomials of a minimal generating set, thus we can derive inequalities on multiplicities. The main result on multiplicities of ideals is the following inequality:

Theorem (See Theorem 4.5). *Let I be a strongly Lech-independent ideal in a local ring $(S,\mathfrak{n})$. Let $x_1,...,x_r$ be a minimal generating set of I such that* $\mathrm{ord}(x_i)=t_i$. *Assume that $t_1\leq t_2\leq...\leq t_r$. Then $e(\mathfrak{n})\leq e(\Gamma)l(S/I)/t_1...t_{d-1}t_d$.*

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Here $e(\Gamma)$ is a constant explained in the sections before Theorem 4.5. Note that if $I = \mathfrak{m}S$ for some flat local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$, then I is strongly Lech-independent and $e(\Gamma) = e(R)$. Also when S is the localization of a standard graded ring over a field at its homogeneous maximal ideal, we get an inequality of the other direction which is a particular case of Lech's conjecture:

Theorem (See Theorem 4.7)**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension of local rings. Suppose S is the localization of a standard graded ring over a field k at its homogeneous maximal ideal, $\mathfrak{m}S$ is the localization of a homogeneous ideal generated by homogeneous elements of degree $t_1 \leq t_2 \leq ... \leq t_r$ and is $\mathfrak{n}$-primary. Then $e(S) \geq e(R)t_1 t_2 \dots t_{r-d}$.*

The paper is organized in the following way. In Section 2 we start with the definition of a standard set, along with some basic definitions and properties on the set of monomials in a polynomial ring. In Section 3 we define strongly Lech-independence and expansion property and prove some equivalent conditions. There are also some examples showing the relation between strongly Lech-independence and other notions. In Section 4 we use strongly Lech-independence to analyze the colength of powers of ideals and derive inequalities on multiplicities.

2. Standard sets in a polynomial ring

Let r be a positive integer, k be a field. Let $P = k[T_1, ..., T_r]$ be a polynomial ring in r variables where T_i's are indeterminates.

Definition 2.1. An ideal I of P is called a *monomial ideal*, if I is generated by monomials. A set of monomials Γ is called a *standard set of monomials*, or a standard set for short, if Γ is a subset of monomials in P such that if u is in Γ, then every monomial dividing u is in Γ.

Let $Mon(\cdot)$ be the set of all the monomials in a polynomial ring or a monomial ideal. For a standard set Γ, let Γ_i be the monomials of degree i in Γ. A standard set is closed under taking factors, hence its complement is closed under taking multiples, which means that the complement is just the set of all monomials in a monomial ideal. Hence we have:

Proposition 2.2. *Γ is a standard set if and only if for some monomial ideal I_Γ, $Mon(P)\backslash\Gamma = Mon(I_\Gamma)$. This builds a bijection between the set of standard sets and the set of monomial ideals in P.*

Let us recall the following basic definition of the graded ring P/I_Γ, where Γ is a standard set. The following definitions can be seen in, for example, [2].

Definition 2.3. Let Γ be a standard set. Let $\underline{z} = (z_1, ..., z_r)$. For a monomial $u = T_1^{a_1} T_2^{a_2} ... T_r^{a_r} \in P$, let $u(\underline{z}) = z_1^{a_1} z_2^{a_2} ... z_r^{a_r}$. The *multigraded Hilbert series* of P/I_Γ is $HS_{P/I_\Gamma}(\underline{z}) = \sum_{u \in \Gamma} u(\underline{z})$. This is a power series in variables $z_1, ..., z_r$. The *Hilbert series* of P/I_Γ is $HS_{P/I_\Gamma}(z) = HS(z, z, ..., z)$. The dimension d of P/I_Γ is the order of $HS_{P/I_\Gamma}(z)$ at the pole $z = 1$; the multiplicity of P/I_Γ is just $\lim_{z \to 1} HS_{P/I_\Gamma}(z)(1-z)^d$.

For convenience sometimes we only care about the standard set Γ, not the monomial ideal I_Γ. So we make the following convention.

Definition 2.4. Let Γ be a standard set. We define the Hilbert series, dimension and multiplicity of Γ to be that of P/I_Γ.

Proposition 2.5. *(Stanley decomposition) For each standard set Γ, there exist a finite set of pairs (u_i, S_i) where every u_i is a monomial in Γ and every S_i is a subset of variables such that $P/I_\Gamma = \oplus_i u_i k[S_i]$ as a k-vector space. In this case, Γ is the disjoint union of $u_i \cdot Mon(k[S_i])$.*

We call such a partition of Γ a *Stanley decomposition* of Γ denoted by (u_i, S_i). The proof of the existence can be seen in [15]. In general, there is a Stanley decomposition for every quotient ring P/I where I is a general ideal and every u_i is an element in P. In [15] we have the following proposition of the Stanley decomposition.

Proposition 2.6. *Let Γ be a standard set with Stanley decomposition (u_i, S_i). Then the multigraded Hilbert series of Γ is $\sum_i \frac{u_i(z)}{\Pi_{T_j \in S_i}(1-z_j)}$. The dimension d of Γ is $max|S_i|$. The multiplicity of Γ is the number of i such that $|S_i| = d$.*

3. Lech-independence and strongly Lech-independence

Throughout the following two sections we make the following assumptions: we assume S is a Noetherian local ring with maximal ideal $\mathfrak{n}$, and I is an ideal of S. We also assume $P = k[T_1, ..., T_r]$ is a polynomial ring in r variables.

Recall that for an element $f \in S$, the order of f, denoted by $\text{ord}(f)$, is the unique integer t such that $f \in \mathfrak{n}^t \backslash \mathfrak{n}^{t+1}$ if $f \neq 0$ and is ∞ if $f = 0$. First we give the definition of Lech-independence in [10] and generalize it to strongly Lech-independence:

Definition 3.1. We say that I is *Lech-independent* if I/I^2 is free over S/I. We say that I is *strongly Lech-independent* if I^i/I^{i+1} is free over S/I for any i. We say that a sequence of elements $x_1, ..., x_r$ is Lech-independent (resp. strongly Lech-independent), if they form a minimal generating set of an ideal which is Lech-independent (resp. strongly Lech-independent).

Obviously, strongly Lech-independence implies Lech-independence.

In [10] we have the following equivalent conditions.

Proposition 3.2. *The following are equivalent for I.*
(1) I is Lech-independent.
(2) Let $\sum_i a_i x_i = 0$ be a relation between the minimal generators x_i of I. Then $a_i \in I$ for all i.
(3) Let ϕ be a presentation matrix for a minimal presentation of the ideal I viewed as an S-module, then ϕ has entries in I.

We have the following equivalent conditions for strongly Lech-independence.

Proposition 3.3. *The following are equivalent for I.*
(1) I is strongly Lech-independent.
(2) $gr_I(S)$ is free over S/I.
(3) $gr_I(S)$ is flat over S/I.

Proof. It suffices to prove (3) $\Rightarrow$ (1). If $gr_I(S)$ is flat over S/I, then for any i, I^i/I^{i+1} is flat over S/I because it is a direct summand of $gr_I(S)$. But it is finitely generated over the local ring S/I, so it is free. So I is strongly Lech-independent by definition. □

We introduce one kind of expansion property for elements in the ring S. For a sequence $x_1, ..., x_r$ of r elements in S and $u = T_1^{a_1} T_2^{a_2} ... T_r^{a_r}$, let $u(x) = x_1^{a_1} x_2^{a_2} ... x_r^{a_r} \in S$. For a monomial ideal $J \subset P$, let $J(x) = (u(x), u \in Mon(J))$. It is an ideal in S.

Definition 3.4. We say a map $\sigma : S/I \to S$ is a *lifting which preserves 0*, or a lifting for short, if $\sigma(0) = 0$ and the composition of σ with the natural quotient map $\pi : S \to S/I$ is the identity map.

Roughly speaking, σ picks a representative for each coset in S/I. We always choose 0 as a representative for simplicity.

Definition 3.5. Let $i < j$ be two positive integers, $x_1, ..., x_r$ be a sequence of r elements in S, I be the ideal $(x_1, ..., x_r)$, Γ a subset of $Mon(P)$. We say $x_1, ..., x_r$ is Γ-*expandable from degree i to j*, if for any lifting $\sigma : S/I \to S$, every element $f \in I^i$ has a unique representation

$$f = \sum_{u \in \Gamma_k, i \le k \le j-1} f_u u(x) \text{ modulo } I^j,$$

such that for any u, $f_u \in \sigma(S/I)$. If S is complete, we say that $x_1, ..., x_r$ is Γ-expandable from degree i to ∞, if for any lifting $\sigma : S/I \to S$ and every element $f \in I^i$ there is a unique representation

$$f = \sum_{u \in \Gamma_k, i \le k} f_u u(x)$$

such that for any u, $f_u \in \sigma(S/I)$. We say that $x_1, ..., x_r$ is Γ-expandable if it is expandable from degree 0 to infinity. The two expressions $f = \sum_{u \in \Gamma_k, i \le k \le j-1} f_u u(x)$ modulo I^j and $f = \sum_{u \in \Gamma_k, i \le k} f_u u(x)$ are called the expansion of f with respect to Γ and the lifting σ, or simply the expansion of f if Γ and σ are clear. We say an ideal is Γ-expandable from degree i to j or ∞ if one minimal generating sequence of the ideal is Γ-expandable from degree i to j or ∞.

From the definition we see that the expansion property depends on the choice of the minimal generators and the order. When we say "an ideal I is Γ-expandable" without pointing out a minimal generating sequence of I which is Γ-expandable, we implicitly choose such a sequence and in this case the notation $u(x), u \in \Gamma$ will make sense. Also when we say $x_1, ..., x_r$ is Γ-expandable for $\Gamma \subset Mon(P)$, we always assume that the length of the sequence r is equal to the number of variables in P.

For the consistency of the notation, we denote $I^\infty = 0$. Note that we always assume S is complete when we talk about "Γ-expandable from degree i to ∞".

Remark 3.6. Suppose $x_1, ..., x_r$ is Γ-expandable from degree i to j, $I = (x_1, ..., x_r)$, and take $f, g \in I^i$ such that $f - g \in I^j$. Then let $f = \sum_{u \in \Gamma_k, i \le k \le j-1} f_u u(x)$ modulo I^j be the unique expansion, we have $g = \sum_{u \in \Gamma_k, i \le k \le j-1} f_u u(x)$ modulo I^j, so the unique expansion of f and g are the same, that is, it only depends on the coset $f + I^j$.

Now we want to relate strongly Lech-independence to some expansion property. We start with two lemmas:

Lemma 3.7. *Let i_1, i_2 be positive integers, i_3 is either a positive integer or the infinity such that $i_1 < i_2 < i_3$. Consider 3 conditions on a sequence $x_1, ..., x_r$.*
(1)$x_1, ..., x_r$ is Γ-expandable from degree i_1 to i_2
(2)$x_1, ..., x_r$ is Γ-expandable from degree i_1 to i_3
(3)$x_1, ..., x_r$ is Γ-expandable from degree i_2 to i_3
Then two of them imply the third one.

Proof. Let $I=(x_1,...,x_r)$. Obviously $u\in\Gamma_k$ implies $u(x)\in I^k$.

Assume (1) and (2) are true, then for any $f\in I^{i_2}\subset I^{i_1}$, by (2) we have

$$f=\sum_{u\in\Gamma_k, i_1\le k\le i_3-1} f_u u(x) \text{ modulo } I^{i_3}.$$

Let

$$f'=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f_u u(x),$$

then $f'=f=0$ modulo I^{i_2}. By (1) the unique expansion of f' modulo I^{i_2} exists and it must be 0. So $f_u=0$ for all $u\in\Gamma_k, i_1\le k\le i_2-1$ and hence we have

$$f=\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} f_u u(x).$$

This shows the existence. The uniqueness just follows from (2) because an expansion from degree i_2 to i_3 can be viewed as an expansion from degree i_1 to i_3 by adding 0's.

Assume (1) and (3) are true. Let $f\in I^{i_1}$, then by (1)

$$f=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f_u u(x)+g,$$

where $g\in I^{i_2}$. By (3),

$$g=\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g_u u(x)+h,$$

where $h\in I^{i_3}$. Thus

$$f=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f_u u(x)+\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g_u u(x)+h$$

is a representation of f. This shows the existence. For uniqueness, let

$$\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f'_u u(x)+\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g'_u u(x)$$

be another representation of f modulo I^{i_3}. Then

$$\begin{aligned} f&=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f_u u(x)+\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g_u u(x)\\ &=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f'_u u(x)+\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g'_u u(x) \text{ modulo } I^{i_3}.\end{aligned}$$

So

$$f=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f_u u(x)=\sum_{u\in\Gamma_k, i_1\le k\le i_2-1} f'_u u(x) \text{ modulo } I^{i_2}.$$

So by (1), $f_u=f'_u$ for any u. Cancelling these terms, we get

$$\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g_u u(x)=\sum_{u\in\Gamma_k, i_2\le k\le i_3-1} g'_u u(x) \text{ modulo } I^{i_3}.$$

By (3) $g_u=g'_u$, which proves the uniqueness.

Assume (2) and (3) are true. Then for any $f\in I^{i_1}$, by (2)

$$f=\sum_{u\in\Gamma_k, i_1\le k\le i_3-1} f_u u(x) \text{ modulo } I^{i_3}.$$

Then
$$f=\sum_{u\in\Gamma_k,i_1\leq k\leq i_2-1} f_u u(x) \text{ modulo } I^{i_2},$$
so the representation exists. Now suppose there is another expression
$$f=\sum_{u\in\Gamma_k,i_1\leq k\leq i_2-1} f'_u u(x)+g, g\in I^{i_2}.$$
Then by (3)
$$g=\sum_{u\in\Gamma_k,i_2\leq k\leq i_3-1} g_u u(x) \text{ modulo } I^{i_3}.$$
So
$$f=\sum_{u\in\Gamma_k,i_1\leq k\leq i_2-1} f'_u u(x)+\sum_{u\in\Gamma_k,i_2\leq k\leq i_3-1} g_u u(x) \text{ modulo } I^{i_3},$$
so $f'_u=f_u$ for any $u\in\Gamma_k, i_1\leq k\leq i_2-1$ by the uniqueness of (2), so the uniqueness of (1) is proved. □

Lemma 3.8. *Let i be an integer. Let $i'_1<i'_2<...$ be a sequence of integers going to infinity and assume that $i<i'_1$. Suppose $x_1,...,x_r$ is Γ-expandable from degree i to i'_j for any j. Then $x_1,...,x_r$ is Γ-expandable from degree i to infinity.*

Proof. Let $I=(x_1,...,x_r)$ and take $f\in I^i$. Let
$$f=\sum_{u\in\Gamma_k,i\leq k\leq i'_j} f_{j,u}u(x)+g_j, g_j\in I^{i'_j}.$$
Suppose $j<j'$. Then
$$\sum_{u\in\Gamma_k,i\leq k\leq i'_j} f_{j,u}u(x)+g_j=\sum_{u\in\Gamma_k,i\leq k\leq i'_{j'}} f_{j',u}u(x)+g_{j'},$$
so
$$\sum_{u\in\Gamma_k,i\leq k\leq i'_j} f_{j,u}u(x)=\sum_{u\in\Gamma_k,i\leq k\leq i'_j} f_{j',u}u(x) \text{ modulo } I^{i'_j}.$$
By the uniqueness of the representation, $f_{j,u}=f_{j',u}$ for any j,j',u. So for any u, $f_{j,u}$ is independent of the choice of j so we can denote it by f_u. The expression $\sum_{u\in\Gamma_k,i\leq k<\infty} f_u u(x)$ makes sense because the ring is complete. We have $f-\sum_{u\in\Gamma_k,i\leq k<\infty} f_u u(x)\in I^{i'_j}$ for any j, so it is 0. So
$$f=\sum_{u\in\Gamma_k,i\leq k<\infty} f_u u(x)$$
is a representation. The uniqueness can be proved modulo $I^{i'_j}$ for any j. □

By the previous two lemmas we are able to describe the relation between strongly Lech-independence and the expansion property.

Proposition 3.9. *The following are equivalent.*
(1) I is strongly Lech-independent.
(2) For every minimal generating sequence $x_1,...,x_r$ of I there is a standard subset Γ of $Mon(P)$ such that I^i/I^{i+1} is free over S/I with basis $u(x)$, with $u\in\Gamma_i$.
(3) For every minimal generating sequence $x_1,...,x_r$ of I there is a standard subset Γ of $Mon(P)$ such that for any i, $x_1,...,x_r$ is Γ-expandable from degree i to $i+1$.

(4) For every minimal generating sequence $x_1, ..., x_r$ of I there is a standard subset Γ of $Mon(P)$ such that for any $i < j$, $x_1, ..., x_r$ is Γ-expandable from degree i to j.
(5) For every minimal generating sequence $x_1, ..., x_r$ of I there is a standard subset Γ of $Mon(P)$ such that for any i, $x_1, ..., x_r$ is Γ-expandable from degree i to infinity.

Proof. (1) implies (2): Let $I = (x_1, ..., x_r)$. Since I^i/I^{i+1} is free, the preimage of a k-basis of $I^i/I^{i+1} \otimes_S S/\mathfrak{n}$ forms an S/I-basis of I^i/I^{i+1}. Consider the special fibre ring $\mathcal{F}_I(S) = gr_I(S) \otimes_S S/\mathfrak{n}$, then it's standard graded over the field $S/\mathfrak{n}$. We may write $\mathcal{F}_I(S) = k[T_1, ..., T_r]/J$ for some homogeneous ideal J such that the image of x_i is $T_i + J$ for $1 \le i \le r$. Let $\Gamma = Mon(k[T_1, ..., T_r]) \backslash Mon(in(J))$, where the initial is taken with respect to any term order which is a refinement of the partial order given by the total degree. Then by the basic propositions of the initial ideal in [1], the monomials in Γ_i is a k-basis of $I^i/I^{i+1} \otimes_S S/\mathfrak{n}$. So taking the preimage, we know that $u(x), u \in \Gamma_i$ is an S/I-basis of I^i/I^{i+1}.

(2) implies (1): trivial.

(2) implies (3): Suppose (2) is true. Let $f \in I^i$. Since I^i/I^{i+1} is generated by $u(x), u \in \Gamma_i$, $f + I^{i+1} = \sum_{u\in\Gamma_i} f_u u(x) + I^{i+1}$. So $f = \sum_{u\in\Gamma_i} f_u u(x) + g, g \in I^{i+1}$. If there is another representation $\sum_{u\in\Gamma_i} f'_u u(x) + g', g \in I^{i+1}$, then in I^i/I^{i+1} we have that $\sum_{u\in\Gamma_i} f'_u u(x) = \sum_{u\in\Gamma_i} f'_u u(x)$. But $u(x), u \in \Gamma_i$ is an S/I-basis, so $f_u = f'_u$ modulo I. But $f_u, f'_u \in \sigma(S/I)$. So $f_u = \sigma(f_u + I) = \sigma(f'_u + I) = f'_u$. This proves (3).

(3) implies (2): Suppose (3) is true. By the existence and the uniqueness of the representation of every element in I^i modulo I^{i+1}, we know that I^i/I^{i+1} is free over S/I with basis $u(x)$, with $u \in \Gamma_i$.

(3) implies (4): use Lemma 3.7 and apply an induction on $|j - i|$.

(4) implies (3): trivial.

(4) implies (5): use Lemma 3.8.

(5) implies (4): use Lemma 3.7 for $i_3 = \infty$. □

Remark 3.10. Let I be a strongly Lech-independent ideal. By Proposition 3.9 I is Γ-expandable for some Γ. So it makes sense to talk about the expansion with respect to such Γ and a lifting σ.

Note that Γ here for which I is expandable is not unique. However, the number of monomials in Γ_i is unique, which is the rank of I^i/I^{i+1} over S/I. This means that $\dim(\Gamma)$ and $e(\Gamma)$ is independent of the choice of Γ. More precisely, we have:

Proposition 3.11. *Let I be a strongly Lech-independent ideal of a local ring $(S, \mathfrak{n})$. Then $\dim(\Gamma)$ and $e(\Gamma)$ are independent of the choice of Γ whenever I is Γ-expandable from degree i to ∞ for any i. If moreover S/I is Artinian, then $\dim(\Gamma) = \dim S$ and $e(I) = l(S/I)e(\Gamma)$. In particular, if I is the maximal ideal $\mathfrak{n}$, then $e(\Gamma) = e(S)$.*

Proof. We know that

$$HS_{P/I_\Gamma}(z) = \sum_{i\ge 0} |\Gamma_i| z^i.$$

Since $|\Gamma_i|$ is independent of the choice of Γ, so is $HS_{P/I_\Gamma}(z)$; and $\dim(\Gamma)$ and $e(\Gamma)$ only depends on $HS_{P/I_\Gamma}(z)$, so they are also independent of the choice of Γ. Now assume S/I is Artinian, we have $\dim S = \dim gr_I(S)$ and $gr_I(S)$ is flat over $S/I = gr_I(S)_0$, so

$$\dim gr_I(S) = \dim S/I + \dim gr_I(S) \otimes_{S/I} S/\mathfrak{n} = \dim \mathcal{F}_I(S).$$

The i-th component of $\mathcal{F}_I(S)$ is $I^i/I^{i+1} \otimes_{S/I} S/\mathfrak{n}$, and

$$\operatorname{rank}_{S/\mathfrak{n}}(I^i/I^{i+1} \otimes_{S/I} S/\mathfrak{n}) = \operatorname{rank}_{S/I} I^i/I^{i+1} = |\Gamma_i|$$

because I^i/I^{i+1} is free over S/I. This means $HS_{P/I_\Gamma}(z) = HS_{\mathcal{F}_I(S)}(z)$ so $\dim P/I_\Gamma = \dim \mathcal{F}_I(S) = \dim S$. Finally,

$$e(I) = \lim_{i\to\infty} (d-1)! l(I^i/I^{i+1})/i^{d-1}$$

and

$$e(P/I_\Gamma) = \lim_{i\to\infty} (d-1)!|\Gamma_i|/i^{d-1}.$$

But $l(I^i/I^{i+1}) = |\Gamma_i| l(S/I)$. So $e(I) = l(S/I)e(\Gamma)$. The last statement is obvious by taking $I = \mathfrak{n}$. □

Proposition 3.12. *Let I be an ideal in S such that I is Γ-expandable for some Γ. Then $T_1, ..., T_r \in \Gamma$.*

Proof. Let $x_1, ..., x_r$ be a sequence of minimal generators of I which is Γ-expandable, then they also form a set of minimal generators of I/I^2. Suppose $T_i \notin \Gamma$. Since Γ is a standard set, it only contains monomials not involving T_i, so expanding x_i uniquely we get $y + z$ where $y \in \sigma(S/I)$ and $z \in (x_1, ..., x_{i-1}, x_{i+1}, ..., x_r)$. Since $y = x_i - z \in I$, $y = 0$. So $x_i \in (x_1, ..., x_{i-1}, x_{i+1}, ..., x_r)$ which is a contradiction because x_i is a minimal generator. □

The following proposition on Lech-independence are taken from [10] by Lech.

Proposition 3.13. *Let $x_1, x_2, ..., x_r$ be Lech-independent in S and $I = (x_1, x_2, ..., x_r)$. Suppose $x_1 = yy'$. Then:*
(1) $y, x_2, ..., x_r$ is Lech-independent.
(2) $I : y = (y', x_2, ..., x_r)$.
(3) There is an exact sequence $0 \to S/(y', x_2, ..., x_r) \xrightarrow{y} S/I \to S/(y, x_2, ..., x_r) \to 0$.
(4) If I is primary to $\mathfrak{n}$, then $l(S/I) = l(S/(y, x_2, ..., x_r)) + l(S/(y', x_2, ..., x_r))$.

Corollary 3.14. Let $x_1, x_2, ..., x_r$ be elements of S and $a_1, ..., a_r$ be positive integers. Suppose $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Lech-independent. Then so is $x_1, ..., x_r$.

There is an analogue of Corollary 3.14 for the expansion property.

Definition 3.15. Let Γ be a standard set. Let $\underline{a} = (a_1, ..., a_r)$ be a set of positive integers. Let Γ' be the following set of monomials $\{u(x_1^{a_1}, ..., x_r^{a_r})x_1^{b_1}x_2^{b_2}...x_r^{b_r} | u \in \Gamma, 0 \le b_i \le a_i\}$. Then Γ' is a standard set. We denote $\Gamma' = \underline{a}\Gamma$.

Remark 3.16. This multiplication on the set of standard sets can be derived from an action on the monomial ideals. Actually, let $\phi_{\underline{a}}$ be an automorphism of P which sends x_i to $x_i^{a_i}$, then $\phi_{\underline{a}}$ maps a monomial to a monomial, hence it extends a monomial ideal to a monomial ideal. Now the multiplication satisfies $I_{\underline{a}\Gamma} = \phi_{\underline{a}}(I_\Gamma)P$. Since the set of actions $\phi_{\underline{a}}, \underline{a} \in \mathbb{N}^r$ is a commutative and associative monoid, the action of $\mathbb{N}^r$ on the set of standard sets is commutative and associative.

Use the notation above, we have the following proposition:

Proposition 3.17. *Let $x_1, x_2, ..., x_r$ be elements of S and $a_1, ..., a_r$ be positive integers. Suppose $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Γ-expandable and Lech-independent. Let $\underline{a} = (a_1, ..., a_r)$, then $x_1, ..., x_r$ is $\underline{a}\Gamma$-expandable.*

Proof. Let $I = (x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r})$ and $J = (x_1, x_2, ..., x_r)$. For any lifting $\sigma : S/J \to S$, we associate a lifting $\sigma' : S/I \to S$: by Lemma 3.18 below every element $f \in S$ has a unique expression $f = \sum_{u \in Mon(P) \setminus Mon((T_1^{a_1}, ..., T_r^{a_r}))} f_u u(x)$ modulo I such that $f_u \in \sigma(S/J)$ for any u. Let $\sigma'(f) = \sum_{u \in Mon(P) \setminus Mon((T_1^{a_1}, ..., T_r^{a_r}))} f_u u(x)$. The image of σ' only depends on the coset $f + I$ and it is a lifting $\sigma' : S/I \to S$. Now $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Γ-expandable, so every element $f \in S$ can be expand uniquely as

$$\sum_{v \in \Gamma} g_v v(x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}) = \sum_{v \in \Gamma, u \in Mon(P) \setminus Mon((T_1^{a_1}, ..., T_r^{a_r}))} g_{u,v} u(x) v(x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r})$$

where $g_v \in \sigma'(S/I)$, $g_{u,v} \in \sigma(S/J)$. As u ranges over $Mon(P) \setminus Mon((T_1^{a_1}, ..., T_r^{a_r}))$ and v ranges over Γ, $u(x)v(x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r})$ ranges over $u(x), u \in \underline{a}\Gamma$, so we are done. □

Lemma 3.18. *Let $x_1, x_2, ..., x_r$ be elements of S and $a_1, ..., a_r$ be positive integers. Suppose $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Lech-independent. Let $P = k[T_1, T_2, ..., T_r]$, $J = (T_1^{a_1}, ..., T_r^{a_r})$, $l = l(S/J) = a_1 a_2 ... a_r$, $I' = (x_1, x_2, ..., x_r)$, $I = J(x) = (x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r})$, . The following holds:*
(1)Every prime filtration of S/J is given by $J = J_l \subset J_{l-1} \subset ... \subset J_0 = S$ such that $J_i/J_{i+1} \cong k$ for any i.
(2)There exist one prime filtration $\mathcal{F}$ given by J_i of S/J such that every J_i is monomial and $J_i(x)/J_{i+1}(x) \cong S/I'$.
(3)Fix one prime filtration $\mathcal{F}$ given by monomial ideals J_i, then there is a one-to-one correspondence between $J_i, 0 \le i \le l-1$ and $Mon(P) \setminus Mon(J)$ which maps J_i to the monomial generator of J_i/J_{i+1}. Denote this map by $M_{\mathcal{F}} : \{0, 1, 2, ..., l-1\} \to Mon(P)$.
(4)For any lifting $\sigma : S/I' \to S$ and $f \in S$ there is a unique expansion modulo I, that is, an equation of the form

$$f = \sum_{u \in Mon(P) \setminus Mon(J)} f_u u(x) \text{ modulo } I$$

such that $f_u \in \sigma(S/I)$.
(5)For any prime filtration $\mathcal{G}$ of J given by monomial ideals J_i, $J_i(x)/J_{i+1}(x) \cong S/I'$.

Proof. (1):The prime filtration always exists for ideals in a Noetherian ring. Since J is $(T_1, ..., T_r)$-primary and $(T_1, ..., T_r)$ is maximal, every factor is $P/(T_1, ..., T_r) \cong k$. The length is l by the definition of length.

(2)Applying Proposition 3.13 inductively we know the following proposition: Let $x_1, x_2, ..., x_{r-1}, x_r^{a_r}$ be Lech-independent, then there exists a filtration of the quotient ring $S/(x_1, x_2, ..., x_{r-1}, x_r^{a_r})$ given by ideals $((x_1, x_2, ..., x_{r-1}, x_r^i)), 0 \le i \le r$ and $((x_1, x_2, ..., x_{r-1}, x_r^i))/((x_1, x_2, ..., x_{r-1}, x_r^{i+1})) \cong S/((x_1, x_2, ..., x_{r-1}, x_r))$. So if $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Lech-independent, we can first get a filtration by changing the power of x_r; then we refine this filtration by changing the power of x_{r-1}; and refine it by changing the power of x_{r-2},...,x_1. Finally we get a filtration such that all the factors are isomorphic, so every factor is isomorphic to the first factor which is $S/(x_1, x_2, ..., x_r)$. Let $<$ be the pure lexicographic order on P with $1 < T_1 < T_2 < ... < T_r$, then this filtration is just of the form $J_i(x)$ where J_i is a monomial generated by $Mon(P)$ except for the largest i monomials not in J. In particular J_i is a prime filtration.

(3)The quotient can be generated by monomials; it is unique because the quotient is k. For every monomial $u \in Mon(P)\backslash Mon(J)$, there is a largest i such that $u \in J_i$, $u \notin J_{i+1}$. So $u \neq 0$ in J_i/J_{i+1}, and since $J_i/J_{i+1} \cong k$, u is a generator of J_i/J_{i+1}.

(4)For any $f \in S$ we find f_u inductively; suppose we have already found f_u for $u = M_{\mathcal{F}}(0), M_{\mathcal{F}}(1), ..., M_{\mathcal{F}}(i-1)$ for $0 \leq i \leq l$ such that

$$f - \sum_{0 \leq j \leq i-1} f_{M_{\mathcal{F}}(j)} M_{\mathcal{F}}(j)(x) \in J_i.$$

This is trivial for $i = 0$ because in this case $f \in J_0 = S$. Now

$$f - \sum_{0 \leq j \leq i-1} f_{M_{\mathcal{F}}(j)} M_{\mathcal{F}}(j)(x) \in g \cdot M_{\mathcal{F}}(i)(x) + J_{i+1}$$

for some $g \in S$. Find the image of $g \cdot M_{\mathcal{F}}(i)(x)$ in $J_i/J_{i+1} \cong S/I' \cdot M_{\mathcal{F}}(i)(x)$; thus

$$(g - \sigma(g+I'))M_{\mathcal{F}}(i)(x) = f - \sum_{0 \leq j \leq i-1} f_{M_{\mathcal{F}}(j)} M_{\mathcal{F}}(j)(x) - \sigma(g+I')M_{\mathcal{F}}(i)(x) \in J_{i+1}.$$

So we find f_u for $u = M_{\mathcal{F}}(0), M_{\mathcal{F}}(1), ..., M_{\mathcal{F}}(i)$ by choosing $f_{M_{\mathcal{F}}(i)} = \sigma(g)$. So by induction we find f_u for $u = M_{\mathcal{F}}(0), M_{\mathcal{F}}(1), ..., M_{\mathcal{F}}(l-1)$ such that

$$f - \sum_{0 \leq j \leq l-1} f_u u(x) \in J_l = J.$$

We claim that an expression of this kind is unique; otherwise we get

$$\sum_{0 \leq j \leq l-1} f_{M_{\mathcal{F}}(j)} M_{\mathcal{F}}(j)(x) = \sum_{0 \leq j \leq l-1} g_{M_{\mathcal{F}}(j)} M_{\mathcal{F}}(j)(x) \text{ modulo } J$$

and $f_{M_{\mathcal{F}}(j)}$, $g_{M_{\mathcal{F}}(j)}$ are not all equal. Find smallest i such that $f_u \neq g_u$ for $u = M_{\mathcal{F}}(i)$. Delete the first i terms we may assume $f_u = 0$ for $u = M_{\mathcal{F}}(j), j < i$. Then take the image in $J_i/J_{i+1} \cong S/I' M_{\mathcal{F}}(i)(x)$ we get $f_{M_{\mathcal{F}}(i)} M_{\mathcal{F}}(i)(x) = g_{M_{\mathcal{F}}(i)} M_{\mathcal{F}}(i)(x)$. So $f_{M_{\mathcal{F}}(i)} = g_{M_{\mathcal{F}}(i)}$ modulo I'. But $f_{M_{\mathcal{F}}(i)}$, $g_{M_{\mathcal{F}}(i)}$ are both liftings by σ of the same coset, so they are equal, which leads to a contradiction. Thus the expansion for every element modulo J is unique.

(5)A generating set of J_i can be given by a generating set of $J_j/J_{j+1}, i \leq j \leq l-1$ and a generating set of $J = J_l$. We know each J_i/J_{i+1} is a quotient of S/I' generated by $M_{\mathcal{G}}(i)(x)$. If this quotient is not faithful, then we get a relation $aM_{\mathcal{G}}(i)(x) = 0$ in J_i/J_{i+1}. Lift $a \neq 0$ to $b = \sigma(a)$, then we get $bM_{\mathcal{G}}(i)(x) \in J_{i+1}$, so there exist $g_u \in \sigma(S/I'), u = M_{\mathcal{G}}(j), i+1 \leq j \leq l-1$ such that

$$bM_{\mathcal{G}}(i)(x) + \sum_{i \leq j \leq l-1} g_{M_{\mathcal{G}}(j)} M_{\mathcal{G}}(j)(x) \in J.$$

But we have another expansion which is $0 \in J$ and $b \neq 0$ because $a \neq 0$, so we get two distinct expansion of 0 modulo J, which leads to a contradiction. □

Here are two typical examples of strongly Lech-independent ideals.

Example 3.19. Suppose I is generated by a regular sequence, or I is the maximal ideal $\mathfrak{n}$, then I is strongly Lech-independent.

Strongly Lech-independence implies Lech-independence, but not conversely by the following example.

Example 3.20. Let S_0 be an Artinian local ring which is not a field and let $\mathfrak{n}_0$ be the maximal ideal of S_0. Let $S = S_0[[x]]/\mathfrak{n}_0 x^2$ and $I = (x)$. Then I is Lech-independent, but not strongly Lech-independent.

Proof. We have $gr_I(S) = S_0[x]/\mathfrak{n}_0 x^2$, $S/I = S_0$, $I/I^2 = S_0 x$ is free over S_0, but $I^2/I^3 = (S_0/\mathfrak{n}_0)x^2$ is not free over S_0. □

There are also some other strongly Lech-independent ideals given by the following proposition:

Proposition 3.21. *Suppose $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local map, and J is a strongly Lech-independent ideal in R. Pick any Γ such that J is Γ-expandable from degree i to ∞ for any i. Such Γ exists by Proposition 3.10. Then $I = JS$ is strongly Lech-independent in S, and I is Γ-expandable from degree i to ∞ for any i. In particular if $J = \mathfrak{m}$, then $I = \mathfrak{m}S$ is strongly Lech-independent. Moreover for any Γ such that $\mathfrak{m}S$ is Γ-expandable from degree i to ∞ for any i, we have $e(\Gamma) = e(R)$.*

Proof. If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat local map, then there is an isomorphism $I^i/I^{i+1} \cong J^i/J^{i+1} \otimes_R S$. Note that freeness and a basis of a module is preserved under any base change. Let $x_1, x_2, ..., x_r$ be a minimal generating set of J, and y_i be the image of x_i, then $y_1, y_2, ..., y_r$ is a minimal generating set of I because the map is local. So if J is Γ-expandable from degree i to ∞ for any i, or equivalently J is Γ-expandable from degree i to $i+1$ for any i, then $u(x), u \in \Gamma_i$ is a basis of J^i/J^{i+1} over R/J. This means $u(y), u \in \Gamma_i$ is a basis of I^i/I^{i+1}. Hence I is Γ-expandable from degree i to $i+1$ for any i, so I is Γ-expandable from degree i to ∞ for any i. If $J = \mathfrak{m}$, We pick a Γ' such that J is Γ'-expandable from degree i to ∞ for any i, then I is also Γ'-expandable from degree i to ∞ for any i. Then $e(\Gamma) = e(\Gamma') = e(R)$ by Proposition 3.11. □

Example 3.22. Let S be a local ring, $x_1, ..., x_r$ be strongly Lech-independent elements in S. Let $S' = S[T]/(T^k - x_1)$. Then S' is flat over S, hence $x_1, ..., x_r$ is still strongly Lech-independent in S'. We will show later that $T, x_2, ..., x_r$ may not be strongly Lech-independent in Example 3.27.

We provide an important source of strongly independent ideals, that is, find a flat local map and extend the maximal ideals of the source ring to the target. We have to be careful that these do not provide all the strongly independent ideals.

Example 3.23. Let k be a field, S be the ring $k[[t, x, y]]/(t^2, x^2 - ty^2)$ and consider the ideal $I = (x, y)$. Then I is strongly independent in S. Let R be the subring generated over k by x, y. Then $R = k[[x, y]]/(x^4)$ and S is not flat over R.

Proof. We have $gr_I(S) = k[t, x, y]/(t^2, x^2 - ty^2)$ as a standard graded ring with $\deg t = 0$, $\deg x = \deg y = 1$. Let $S_0 = gr_I(S)_0 = k[t]/t^2$. Now $gr_I(S)_1 = S_0 x + S_0 y$ is free over S_0. For $i \geq 2$,

$$gr_I(S)_i = \sum_{0 \leq j \leq i} S_0 x^j y^{i-j} / \sum_{2 \leq j \leq i} S_0 (x^j y^{i-j} - t x^{j-2} y^{i-j+2}).$$

Note that the set $\{x^j y^{i-j} - t x^{j-2} y^{i-j+2}\}$ is part of a minimal basis of the free module $\sum_{0 \leq j \leq i} S_0 x^j y^{i-j}$, so the quotient is still a free S_0-module, which implies that I is strongly Lech-independent. Let $\phi : k[[x, y]] \to S$. Then $R = k[[x, y]]/\ker \phi$ and $\ker \phi = (t^2, x^2 - ty^2) \cap k[[x, y]]$. Let $<$ be the pure lexicographic order such that $1 > t > x > y$. Then for a power series $f \in k[[t, x, y]]$, $f \in k[[x, y]]$ if and only if the largest term of f is in $k[[x, y]]$. We apply the Buchberger's algorithm to compute the ideal of largest terms. The Gröbner basis of the ideal $(t^2, x^2 - ty^2)$ is $t^2, x^2 - ty^2, x^4$, so $(t^2, x^2 - ty^2) \cap k[[x, y]] = (x^4)$. So $R = k[[x, y]]/(x^4)$. Now S has a minimal generating set $1, t$ as an R-module and a nontrivial relation $x^2 - ty^2 = 0$, so S is not free over R. $\square$

If we allow a change in the residue field we have another example. The proof is similar to Example 3.23.

Example 3.24. Let $S = \mathbb{C}[[x_1, x_2]]/(x_1^2 + \sqrt{-1}x_2^2)$ and $R = \mathbb{R}[[x_1, x_2]]/(x_1^4 + x_2^4)$. Then R is a subring of S, $\mathfrak{m}S = (x_1, x_2)S$ is strongly independent, and S is not flat over R.

Corollary 3.14 is a strong proposition for Lech-independence. It allows us to replace Lech-independent elements with their roots. However, its converse does not hold, so in general we cannot replace elements with their powers while preserving independence property. In fact, the condition "stays Lech-independent after raising to any power" is a strong condition which is equivalent to being a regular sequence. To be precise, we have the following equivalent conditions:

Proposition 3.25. *Let I be an ideal of a complete local ring S which contains a field k, and $x_1, ..., x_r$ be a set of minimal generators. Then the following are equivalent.*
(1) For any positive integer $a_1, ..., a_r$, $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Lech-independent.
(2) For any positive integer $a_1, ..., a_r$, $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is strongly Lech-independent.
(3) $x_1, ..., x_r$ is $Mon(P)$-expandable.
(4) $x_1, ..., x_r$ forms a regular sequence.

Proof. (2) implies (1) is trivial.

(1) implies (3): Let $a = min\{a_i\}$. We claim that I^i/I^{i+1} is free with rank equal to $\dim_k P_i$ for $i < a$. Let $I = (x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r})$, $I_1 = (x_1, ..., x_r)$, $J = (T_1^{a_1}, T_2^{a_2}, ..., T_r^{a_r})$, $J_1 = (T_1, ..., T_r)$. Consider a filtration $J_i = J + J_1^i$, and $I_i = J_i(x)$ for $0 \leq i \leq \sum_i a_i$. P/J has length $l = a_1 a_2 ... a_r$. Thus $\sum_i l(J_i/J_{i+1}) = l$. We can refine the filtration $\mathcal{F} = J_i$ by $\mathcal{G} = K_j$ such that every K_j is a monomial ideal and $K_j/K_{j+1} \cong k$ for every j. By Lemma 3.18, $K_j(x)/K_{j+1}(x) \cong S/I$. So $J_i(x)/J_{i+1}(x)$ has a filtration such that each factor is of the form $K_j(x)/K_{j+1}(x)$ which is free, thus $J_i(x)/J_{i+1}(x)$ is free. The number of factors is just the length of J_i/J_{i+1}, so it is free of rank $l(J_i/J_{i+1})$. The set $\{u(x), u \in J_i \backslash J_{i+1}\}$ is a generator of J_i/J_{i+1} and by comparing the cardinality this set is minimal, so it is a free basis. In particular if $i < a = min\{a_i\}$, then $J_i = J_1^i$, $I_i = J_i(x) = I^i$, $J_{i+1} = J_1^{i+1}$, $I_{i+1} = J_{i+1}(x) = I^{i+1}$ and $l(J_i/J_{i+1}) = \dim_k P_i$. So I^i/I^{i+1} is free with rank equal to $dim_k P_i$. As we let $a \to \infty$, we know that this is true for all i; thus I is strongly Lech-independent which is $Mon(P)$-expandable.

(3) implies (4): take an element $\bar{f} \in S/(x_1, ..., x_j)$ for some j. Let f be a preimage in S. Suppose $x_{j+1} f \in (x_1, ..., x_j)$. We expand $f = \sum f_u u(x)$. Then $x_{j+1} f = \sum f_u \cdot (uT_{j+1})(x)$. This expansion satisfies $f_u \in \sigma(S/(x_1, ..., x_r))$, so it must be the unique expansion. We claim that for any $g \in (x_1, ..., x_j)$ with an expansion $\sum g_u u(x)$, $g_u \neq 0$ only if $u \in (T_1, ..., T_j)$. We may choose the map σ such that it is k-linear, so in particular, additive; let $g = \sum_{1 \leq i \leq j} g_i x_i$. Consider

the expansion of g_i which is $\sum_{u\in Mon(P)} g_{i,u}u(x)$; then

$$g = \sum_{1\le i\le j, u\in Mon(P)} g_{i,u}(uT_i)(x) = \sum_{1\le i\le j, u/T_i\in Mon(P)} g_{i,u/T_i}u(x).$$

But fixing u,

$$\sum_{1\le i\le j, u/T_i\in Mon(P)} g_{i,u/T_i} \in \sigma(S/(x_1,...,x_r))$$

because σ is k-linear, so $\sum_{1\le i\le j, u/T_i\in Mon(P)} g_{i,u/T_i}u(x)$ is an expansion of g, so it must be the unique expansion, and in this expansion $\sum_{1\le i\le j, u/T_i\in Mon(P)} g_{i,u/T_i} \neq 0$ only if $u \in (T_1,...,T_j)$. Now apply the claim to $x_{j+1}f$, we see that $f_u \neq 0$ implies $uT_{j+1} \in (T_1,...,T_j)$, so $u \in (T_1,...,T_j)$ and in this case $u(x) \in (x_1,...,x_j)$, so $f \in (x_1,...,x_j)$. Since this is true for any j, we get (4).

(4) implies (2): if $x_1,...,x_r$ forms a regular sequence, then $gr_{(x_1,...,x_r)}S \cong S/(x_1,...,x_r)[T_1,...,T_r]$, so we get (2). □

Remark 3.26. The proof of Kunz's theorem in [8] uses the equivalence of (1) and (4) in the previous proposition. To be precise, suppose R is a local ring of positive characteristic p such that the Frobenius action on R is flat. let $x_1,...,x_r$ be a minimal generating set of the maximal ideal of R. Then it is Lech-independent. So after a flat base change F^e it is still strongly Lech-independent. But after a flat base change the ideal just becomes $x_1^{p^e}, x_2^{p^e}, ..., x_r^{p^e}$. As e goes to infinity we know that any power of $x_1,...,x_r$ is Lech-independent. So $x_1,...,x_r$ forms a regular sequence, hence the ring is regular. The above proof can also be seen in standard textbooks or lecture notes, for instance, [14].

Let $x_1,x_2,...,x_r$ be elements of S and $a_1,...,a_r$ be positive integers. Let $I = (x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}), I' = (x_1,x_2,...,x_r)$. In the above paragraph we know I is strongly Lech-independent implies that I' is Γ-expandable for some Γ. Also I is Lech-independent implies that I' is Lech-independent. So it is natural to ask whether I is strongly Lech-independent implies that I' is strongly Lech-independent, and by Proposition 3.17 it suffices to prove the following: I' is Γ-expandable implies I' is Γ-expandable from degree i to infinity for any i. However, both implications are wrong. This is the reason to introduce the complicated notion "Γ-expandable from degree i to j" to describe strongly Lech-independence.

Example 3.27. Let $S = k[[x,y,t]]/(t^2, ty^2 - x^8)$ and $I = (x,y)$, $I' = (x^4,y)$. Let $P = k[T_1,T_2], \Gamma = Mon(P)\backslash Mon((T_1^8))$. Then I' is strongly Lech-independent; I is Γ-expandable, but it is not strongly Lech-independent. In particular for some i I is not Γ-expandable from degree i to infinity.

Proof. For the first part, let $S' = k[[X,y,t]]/(t^2, ty^2 - X^2)$ and $J = (X,y)$. Then by Example 3.23 J is strongly Lech-independent. There is a map $S' \to S : X \to x^4, y \to y, t \to t$ and it is flat local. So $I' = JS$ is strongly Lech-independent. For the rest part, we may choose a local monomial order $<$ on S such that the initial ideal of $K = (t^2, ty^2 - x^8)$ is (t^2, x^8). Here the initial of an element is the smallest term in that element and the initial ideal is the ideal generated by smallest terms of elements in an ideal. For example, choose $<$ to be the pure lexicographic order on x,y,t such that $x < y < t < 1$. Then the initial of the two generator is t^2 and x^8, and they are relatively prime, so they form a Gröbner basis of K. So every element

f in $S = k[[x,y,t]]/I$ can be expressed uniquely as a (possibly infinite) sum

$$f = \sum_{i=0,1,0\leq j\leq 7,k} f_{i,j,k}t^i x^j y^k = \sum_{0\leq j\leq 7,k} (f_{0,j,k} + tf_{1,j,k})x^j y^k.$$

Also $S/I = k[t]/t^2$ so we may choose the lifting $\sigma : S/I \to S$ which maps $a + bt + I$ to $a+bt$ for any $a, b \in k$. Then we know I is Γ-expandable by the unique expression of f. However, I^2/I^3 is minimally generated by x^2, xy, y^2 with a nontrivial relation $ty^2 = 0$, so it is not free over S/I, so I is not strongly Lech-independent. □

Example 3.28. Let $S = k[[x,y,t]]/(t^2, ty - x^2)$, $I = (x, y)$, $P = k[T_1, T_2]$, and $\Gamma = Mon(P)\backslash Mon((T_1^2))$. By the same token above I is Γ-expandable, I/I^2 is minimally generated by x, y with $ty = 0$, so it is not free over S/I, so I is not Lech-independent. So being Γ-expandable does not imply Lech-independence.

There is a special implication; being strongly independent implies being Ratliff-Rush.

Definition 3.29. Let S be a local ring, I an ideal of S. Then $\widetilde{I} = \cup_i I^{i+1} : I^i$ is called the Ratliff-Rush closure of I. We say that I is Ratliff-Rush if its Ratliff-Rush closure is itself.

Now the following proposition is trivial.

Proposition 3.30. *I is Ratliff-Rush if and only if $Ann_{S/I}(I^i/I^{i+1}) = 0$ for any i. In particular, strongly independence implies being Ratliff-Rush.*

Remark 3.31. The converse of Proposition 3.30 does not hold. For example, Consider $S = k[[t_1, t_2, x, y]]/(t_1^2, t_2^2, t_1t_2, t_1x^2 - t_2y^2)$ and $I = (x, y)$. Then I is Lech-independent. S/I,I/I^2 is free over S/I. For $i \geq 2$,

$$I^i/I^{i+1} = \sum (S/I)x^j y^{i-j}/\sum k \cdot t_1x^j - t_2x^{j-2}y^2.$$

The set $x^j y^{i-j}$ is a minimal generating set, but not a basis, so I^i/I^{i+1} is not free, so I is not strongly Lech-independent. We claim that I is Ratliff-Rush. It suffices to prove $\mathrm{Ann}_{S/I}(x^i + I^{i+1}) = 0$. Suppose this is not true, then there exist $a, b, c \in k$ not all 0 such that $(a + bt_1 + ct_2)x^i = 0$ in I^i/I^{i+1}. Equivalently, there exist $a_j \in k$ such that $(a + bt_1 + ct_2)x^i + \sum a_j(t_1x^j - t_2x^{j-2}y^2) = 0$ in $k[[t_1, t_2, x, y]]/(t_1^2, t_2^2, t_1t_2)$. But the elements $\{t_1x^j - t_2x^{j-2}y^2, x^i, t_1x^i, t_2x^i\}$ are k-linearly independent in $k[[t_1, t_2, x, y]]/(t_1^2, t_2^2, t_1t_2)$, thus $a = b = c = 0$, which is a contradiction.

4. Strongly Lech-independence and inequalities on multiplicities of ideals

Throughout this section, we keep the same assumptions as the last section. We also assume that $(S, \mathfrak{n})$ is a complete local ring with a coefficient field k unless otherwise stated. We begin with a lemma which is a reformulation of the expansion property.

Lemma 4.1. *Let Γ be a standard set, I be an ideal in S which is $\mathfrak{n}$-primary and Γ-expandable. Take $f_1, f_2, ..., f_l \in S$ such that their images in S/I form a k-basis of S/I, and define a k-linear map $\sigma : S/I \to S$ which maps $f_i + I$ to f_i. Then σ*

is a lifting, and expanding f as a linear combination of $f_i \cdot u(x)$ gives a k-linear isomorphism

$$S \cong \prod_{1\leq i\leq l, u\in\Gamma} k\cdot f_i u(x).$$

Proof. Since σ is k-linear, $\sigma(0)=0$. Every element in S/I is of the form $\sum_{1\leq i\leq l} a_i f_i + I$ for $a_1,...,a_l \in k$. Let $\pi : S \to S/I$ be the projection, then $\pi\sigma(\sum_{1\leq i\leq l} a_i f_i + I) = \pi(\sum_{1\leq i\leq l} a_i f_i) = \sum_{1\leq i\leq l} a_i f_i + I$. So σ is a lifting. For every $f\in S$, $f=\sum_u f_u u(x)$. We write $f_u = \sum_i c_{i,u} f_i$ modulo I for $c_{i,u}\in k$. But σ is k-linear, so $\sum_i c_{i,u} f_i \in \sigma(S/I)$, so $f_u = \sum_i c_{i,u} f_i$ in S. So $f=\sum_u f_u u(x) = \sum_{i.u} c_{i,u} f_i u(x)$. This defines the map, and it is well-defined by the uniqueness of the expansion. The map is surjective since the preimage of an expansion is just the value of the sum, and it exists when S is complete. It is injective because if two elements give the same expansion then they are both equal to the sum, hence they must be equal. It suffices to prove linearity. Since $f_i + I$ is a k-basis of the k-vector space S/I, we can define $\sigma : \sum_i c_i(f_i+I) \to \sum_i c_i f_i, c_i\in k$. Then σ is a well-defined k-linear map which lifts the identity. Now suppose $f=\sum_u f_u u(x)$, $g=\sum_u g_u u(x)$, $c\in k$. Then $f+cg=\sum_u (f_u+cg_u)u(x)$. By the assumption on the expansion $f_u=\sigma(f_u+I), g_u=\sigma(g_u+I)$, so $f_u+cg_u=\sigma(f_u+cg_u+I)$. So f_u+cg_u is in $\sigma(S/I)$. So $f+cg=\sum_u(f_u+cg_u)u(x)$ is the unique expansion of $f+cg$. This proves the lemma. □

Corollary 4.2. With the same notation as in Lemma 4.1, let t be a positive integer. Set

$$A_{1,t} = \{f_i u(x) | 1\leq i\leq l, u\in\Gamma, f_i u(x)\notin \mathfrak{n}^t\},$$

and

$$A_{2,t} = \{f_i u(x) | 1\leq i\leq l, u\in\Gamma, \operatorname{ord}(f_i)+\sum \operatorname{ord}(x_j)\deg_{T_j}(u) < t\}.$$

Then we have:
(1) $S/\mathfrak{n}^t$ can be spanned over k by $A_{1,t}$.
(2) $A_{1,t}\subset A_{2,t}$. So $S/\mathfrak{n}^t$ can be spanned by $A_{2,t}$.
(3) If the set $A_{2,t}$ is linearly independent modulo $\mathfrak{n}^t$, then its image form a basis of $S/\mathfrak{n}^t$. So $\dim_k S/\mathfrak{n}^t = |A_{2,t}|$.

Proof. Every element in $S/\mathfrak{n}^t$ has the form $f+\mathfrak{n}^t$, and we can represent f as $f=\sum_{1\leq i\leq l, u\in\Gamma} c_{i,u} f_i u(x)$ by the unique expansion property. Since $I\neq S$, $I^t\subset\mathfrak{n}^t$. So $u\in\Gamma_j, j\geq t$ implies $u(x)\in I^t\subset\mathfrak{n}^t$. So $f=\sum_{1\leq i\leq l, u\in\Gamma_j, j<t} c_{i,u} f_i u(x)$ in $S/\mathfrak{n}^t$ and this is a finite linear combination. So $f+\mathfrak{n}^t$ is in the span of all the $f_i u(x)$, so it's in the span of $f_i u(x)$ such that $f_i u(x)\notin\mathfrak{n}^t$ because $f_i u(x)\in\mathfrak{n}^t$ means that $f_i u(x)=0$ in $S/\mathfrak{n}^t$. This proves (1). For the second claim, note that if $f_i u(x)\notin A_{2,t}$, then $\operatorname{ord}(f_i)+\sum_j \operatorname{ord}(x_j)\deg_{T_j}(u)\geq t$, so $\operatorname{ord}(f_i u(x))\geq t$, $f_i u(x)\in\mathfrak{n}^t$, and $f_i u(x)\notin A_{1,t}$. This proves (2). (3) is obvious by (2). □

Recall that the Hilbert series of S is $HS_S(z)=\sum_{i\geq 0} dim_k(\mathfrak{n}^i/\mathfrak{n}^{i+1})z^i$. Define a partial order $\leq$ on $\mathbb{R}[[z]]$ to be degreewise comparison, that is, $\sum_{i\geq 0} a_i z^i \leq \sum_{i\geq 0} b_i z^i$ if $a_i\leq b_i$ for all i. We have an embedding $\mathbb{R}[z]_{(a-z)}\hookrightarrow\mathbb{R}[[z]]$ for any $a\neq 0$. That means if $z=0$ is not a pole of a rational series $a(z)$ then we can view $a(z)$ as an element in $\mathbb{R}[[z]]$, while at the same time $a(z)$ is defined over $\mathbb{C}$ except for finitely many poles of $a(z)$, so we can take limits in $\mathbb{C}$.

Lemma 4.3. *Let d be a positive integer, $a(z)=\sum_{i\geq 0} a_i z^i$ be a rational series satisfying the following properties:*
(P1) $a(z)$ only has poles at roots of unity;
(P2$_d$) $z=1$ is a pole of $a(z)$ with order d;
(P3$_d$) The orders of poles of $a(z)$ except for 1 are less than d.
Then we have

$$\lim_{z\to 1}\sum_{i\geq 0} a_i z^i (1-z)^d = \lim_{k\to\infty} \frac{(d-1)!}{(d+k-1)!}\frac{\partial^k a(0)}{\partial z^k}. \tag{4.1}$$

Proof. We can express $a(z)$ using partial-fraction decomposition. To be precise, let U be the set of poles of $a(z)$, then there exist finitely many real numbers $e_{i,\xi}, 1\leq i\leq d-1, \xi\in U$, a real number $e_0\neq 0$, and a polynomial $b(z)$ such that

$$a(z) = \sum_{1\leq i\leq d-1,\xi\in U} e_{i,\xi}(\xi - z)^{i-d} + e_0(1-z)^{-d} + b(z). \tag{4.2}$$

Let L be the map $a(z)\to \lim_{k\to\infty}\frac{(d-1)!}{(d+k-1)!}\frac{\partial^k a(0)}{\partial z^k}$. Then it is additive. We apply L to each term in the right side of (4.2). If $1\leq i\leq d-1$,

$$L((\xi-z)^{i-d}) = \lim_{k\to\infty}\frac{(d-1)!(d-i+k-1)!}{(d-i-1)!(d+k-1)!}(\xi-0)^{i-d-k} = 0$$

as $(\xi-0)^{i-d-k}$ is bounded and $\frac{(d-1)!(d-i+k-1)!}{(d-i-1)!(d+k-1)!}$ goes to 0,

$$L((1-z)^{-d}) = \lim_{k\to\infty}\frac{(d-1)!(d+k-1)!}{(d-1)!(d+k-1)!}(1-0)^{i-d-k} = 1,$$

and $L(b(z))=0$ as $b(z)$ is a polynomial. This means the right side of (4.1) is $L(a(z))=e_0$. The left side is also e_0, so they are equal. □

Lemma 4.4. *Let $\sum_{i\geq 0} a_i z^i, \sum_{i\geq 0} b_i z^i$ be two rational series satisfying (P1), (P2$_d$) and (P3$_{d+1}$). Assume*

$$\sum_{i\geq 0} a_i z^i/(1-z) \leq \sum_{i\geq 0} b_i z^i/(1-z),$$

then

$$\lim_{z\to 1}\sum_{i\geq 0} a_i z^i(1-z)^d \leq \lim_{z\to 1}\sum_{i\geq 0} b_i z^i (1-z)^d.$$

Proof. Let $\sum_{i\geq 0} a'_i z^i = \sum_{i\geq 0} a_i z^i/(1-z), \sum_{i\geq 0} b'_i z^i = \sum_{i\geq 0} b_i z^i/(1-z)$. It suffices to prove that

$$\lim_{z\to 1}\sum_{i\geq 0} a'_i z^i (1-z)^{d+1} \leq \lim_{z\to 1}\sum_{i\geq 0} b'_i z^i (1-z)^{d+1}. \tag{4.3}$$

Now $\sum_{i\geq 0} a'_i z^i$ is a rational series satisfying (P1), (P2$_{d+1}$) and (P3$_{d+1}$) , so by Lemma 4.3 the limit on the left side of (4.3) is equal to $\lim_{k\to\infty}\frac{d!}{(d+k)!}\frac{\partial^k a'(0)}{\partial z^k}$, and similar for the right side. Now the partial order on the power series is preserved by taking derivatives, multiplying a positive constant, and evaluate at 0. So

$$\frac{d!}{(d+k)!}\frac{\partial^k a'(0)}{\partial z^k} \leq \frac{d!}{(d+k)!}\frac{\partial^k b'(0)}{\partial z^k},$$

and take the limit when $k\to\infty$. □

Theorem 4.5. *Let I be an ideal in S which is $\mathfrak{n}$-primary, $x_1, ..., x_r$ be a minimal generating sequence of I such that the order of x_i is t_i and $t_1 \leq t_2 \leq ... \leq t_r$. Denote $d = \dim S$. Pick Γ such that $x_1, ..., x_r$ is Γ-expandable. We choose f_i such that their images form a homogeneous k-basis of $gr_{\mathfrak{n}}(S/I)$.*
(1) Let $c(z) = \sum_{t\geq 0} c_t z^t$, where c_t is the number of $f_i u(x)$ such that $1 \leq i \leq l, u \in \Gamma, \operatorname{ord}(f_i) + \sum_j \operatorname{ord}(x_j)\deg_{T_j}(u) = t$. Then

$$c(z) = HS_{S/I}(z)HS_\Gamma(z^{t_1}, z^{t_2}, ..., z^{t_r})$$

and $c(z)$ satisfies (P1), (P2$_d$), (P3$_{d+1}$).
(2) We have

$$HS_S(z)/(1-z) \leq c(z)/(1-z).$$

If for any t, the set

$$A_{2,t} = \{f_i u(x) | 1 \leq i \leq l, u \in \Gamma, \operatorname{ord}(f_i) + \sum_j \operatorname{ord}(x_j)\deg_{T_j}(u) < t\}$$

is k-linearly independent modulo $\mathfrak{n}^t$, then

$$HS_S(z)/(1-z) = c(z)/(1-z).$$

(3) We have:

$$l(S/I)e(\Gamma)/t_r t_{r-1}...t_{r-d+1} \leq \lim_{z\to 1} c(z)(1-z)^d \leq l(S/I)e(\Gamma)/t_1 t_2 ... t_d.$$

(4) There is an upper bound of the multiplicity of the maximal ideal:

$$e(\mathfrak{n}) \leq e(\Gamma)l(S/I)/t_1...t_{d-1}t_d.$$

If moreover the set

$$A_{2,t} = \{f_i u(x) | 1 \leq i \leq l, u \in \Gamma, \operatorname{ord}(f_i) + \sum_j \operatorname{ord}(x_j)\deg_{T_j}(u) < t\}$$

is k-linearly independent modulo $\mathfrak{n}^t$ for any t, then there is also a lower bound:

$$e(\mathfrak{n}) \geq e(\Gamma)l(S/I)/t_r t_{r-1}...t_{r-d+1}.$$

Proof. (1) By definition,

$$c(z) = \sum_{i,u} z^{\operatorname{ord}(f_i)+\sum_j \operatorname{ord}(x_j)\deg_{T_j}(u)}$$

$$= \sum_i z^{\operatorname{ord}(f_i)} \sum_{u\in\Gamma} z^{\sum \operatorname{ord}(x_j)\deg_{T_j}(u)}$$

$$= HS_{S/I}(z)\sum_{u\in\Gamma} u(z^{t_1}, z^{t_2}, ..., z^{t_r})$$

$$= HS_{S/I}(z)HS_\Gamma(z^{t_1}, z^{t_2}, ..., z^{t_r}).$$

Let (u_i, S_i) be a Stanley decomposition of Γ. Then $HS_\Gamma(\underline{z}) = \sum_i \frac{u_i(z)}{\Pi_{T_j\in S_i}(1-z_j)}$. So

$$c(z) = HS_{S/I}(z)\sum_i \frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{\Pi_{T_j\in S_i}(1-z^{t_j})}. \tag{4.4}$$

The right side of (4.4) has two factors. The first factor $HS_{S/I}(z)$ is a polynomial with $HS_{S/I}(1) = l(S/I) \geq 0$, so it's regular near $z = 1$. The other factor is a finite sum, and we compute the order of each term in the sum. Note that

$$\frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{\Pi_{T_j \in S_i}(1 - z^{t_j})} = \frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{(\Pi_{T_j \in S_i}(1 + z + ... + z^{t_j - 1}))(1 - z)^{|S_i|}},$$

so the order at $z = 1$ of the i-th term is just $|S_i|$, and the other poles are given by t_j-th roots of unity; every t_j-th root of unity is a single pole of $1/(1+z+...+z^{t_j-1})$, so the order of the i-th term at the other poles is at most $|S_i|$. So the order of the sum at $z = 1$ is $max|S_i|$, which is just d, and the order of the sum at the other poles is at most d. This means that $c(z)$ satisfies (P1), (P2$_d$), (P3$_{d+1}$).

(2) If the images of f_i's form a homogeneous k-basis of $gr_{\mathfrak{n}}(S/I)$ then f_i's form a k-basis of S/I. The $(t-1)$-th coefficient of $HS_S(z)/(1-z)$ is the sum of the coefficients of $1, z, ..., z^{t-1}$ in $HS_S(z)$, which is $l(S/\mathfrak{n}^t)$. The $(t-1)$-th coefficient of $c(z)/(1-z)$ is the sum of the coefficients of $1, z, ..., z^{t-1}$ in $c(z)$, so it is the number of $f_i u(x)$ such that $\operatorname{ord}(f_i) + \sum_j \operatorname{ord}(x_j) \deg_{T_j}(u) < t$, which is $|A_{2,t}|$. It is no less than the length of $S/\mathfrak{n}^t$ by Corollary 4.2, and the equality holds if the assumption of (2) holds. So $HS_S(z)/(1-z) \leq c(z)/(1-z)$, and the equality holds if for any t, $A_{2,t}$ is k-linearly independent modulo $\mathfrak{n}^t$.

(3) By the proof in (1),

$$\lim_{z\to 1} c(z)(1-z)^d = \lim_{z\to 1} HS_{S/I}(z) \sum_i \frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{\Pi_{T_j \in S_i}(1 - z^{t_j})}(1-t)^d$$

$$= l(S/I) \lim_{z\to 1} \sum_i \frac{u_i(1, 1, ..., 1)}{\Pi_{T_j \in S_i}(1 - z^{t_j})}(1-z)^d$$

$$= l(S/I) \sum_{i,|S_i|=d} \frac{u_i(1, 1, ..., 1)}{\Pi_{T_j \in S_i} t_j} = l(S/I) \sum_{i,|S_i|=d} \frac{1}{\Pi_{T_j \in S_i} t_j}.$$

Also $e(\Gamma) = \sum_{i,|S_i|=d} 1$. By the choice of $t_1, ..., t_r$,

$$t_1 t_2 ... t_d \leq \Pi_{T_j \in S_i} t_j \leq t_{r-d+1} ... t_{r-1} t_r$$

whenever $|S_i| = d$. So

$$1/t_1 t_2 ... t_d \geq 1/\Pi_{T_j \in S_i} t_j \geq 1/t_{r-d+1} ... t_{r-1} t_r.$$

Take the sum over i where $|S_i| = d$ and multiply by $l(S/I)$, we get the conclusion.

(4) By Lemma 3.11 $\dim S = \dim \Gamma = d$, so $e(\mathfrak{n}) = \lim_{z\to 1} HS_S(z)(1-z)^d$. In (2) we get $HS_S(z)/(1-z) \leq c(z)/(1-z)$. $c(z)/(1-z)$ satisfies (P1), (P2$_{d+1}$) and (P3$_{d+1}$) by (1); $HS_S(z)/(1-z)$ has a single pole at $z = 1$ of order $d + 1$ so it also satisfies (P1), (P2$_{d+1}$) and (P3$_{d+1}$). So we can apply Lemma 4.4 to get

$$\lim_{z\to 1} HS_S(z)(1-z)^d \leq \lim_{z\to 1} c(z)(1-z)^d \leq l(S/I)e(\Gamma)/t_1 t_2 ... t_d.$$

So the first claim is true. For the second part, we have $HS_S(z)/(1-z) = c(z)/(1-z)$ so

$$\lim_{z\to 1} HS_S(z)(1-z)^d = \lim_{z\to 1} c(z)(1-z)^d \geq e(\Gamma) l(S/I)/t_r t_{r-1} ... t_{r-d+1}.$$

□

The condition in Theorem 4.5(4) is quite strong and is false in general. However, it can be satisfied in the standard graded case. The following lemma builds a relation between the standard graded case, the local case and the complete local case.

Proposition 4.6. *Let $(S_g, \mathfrak{n}_g)$ be a standard graded ring over a field k, $(S_L, \mathfrak{n}_L)$ be the localization of $(S_g, \mathfrak{n}_g)$ at $\mathfrak{n}_g$, and the completion of $(S_L, \mathfrak{n}_L)$ is $(S, \mathfrak{n})$. Let I_g be a homogeneous ideal in S_g, and let $I_L = I_g S_L, I = I_g S$. Choose a set of homogeneous minimal generators $y_1, ..., y_e$ of $\mathfrak{n}_g$. Then:*
(1) $S_g = k[y_1, \dots, y_e]/I_g$, $S_L = k[y_1, \dots, y_e]_{(y_1,\dots,y_e)}/I_L$, $S = k[[y_1, \dots, y_e]]/I$.
(2) We have embeddings of rings $S_g \overset{i_1}{\hookrightarrow} S_L \overset{i_2}{\hookrightarrow} S$. More generally, for any S_g-ideal J we have injections $S_g/J \hookrightarrow S_L/JS_L \hookrightarrow S/JS$.
(3) Either I_g, I_L, I are all Artinian or none of them is Artinian.
(4) Assume that I_g, I_L, I are all Artinian, then for any t, $I_g^t/I_g^{t+1} \cong I_L^t/I_L^{t+1} \cong I^t/I^{t+1}$ where these isomorphisms are induced by i_1 and i_2.
(5) Assume that I_L, I are both Artinian, then either I_L, I are both strongly Lech-independent or none of them is strongly Lech-independent. If they are strongly Lech-independent and one of them is Γ-expandable from degree i to j for any $i < j$, then both of them are Γ-expandable from degree i to j for any $i < j$.
(6) The notion $\operatorname{ord}(f)$ is well-defined for nonzero elements f in S_g, S_L, S and the different orders are compatible via i_1 and i_2.
(7) If I_g, I_L, I are all Artinian then $e(I_g) = e(I_L) = e(I)$. In particular $e(\mathfrak{n}_g, S_g) = e(S_L) = e(S)$.

Proof. (1) is trivial. To prove (2) it suffices to prove that for any ideal J of S_g, $S_g/J \hookrightarrow S/JS$. This is true because the map is faithfully flat, hence injective. We know the dimension of a standard graded ring over a field k is equal to the dimension of its localization at the homogeneous maximal ideal, and the dimension of a local ring is equal to the dimension of its completion. This implies that $\dim S_g/I_g = \dim S_L/I_L = \dim S/I$, so either they are all 0 or they are all nonzero, which implies (3). To prove (4), note that i_1, i_2 are both faithfully flat extensions. So $I_L^t/I_L^{t+1} \cong I_g^t/I_g^{t+1} \otimes_{S_g} S_L \cong I_g^t/I_g^{t+1} \otimes_{S_g/I_g} S_L/I_L$ and $I^t/I^{t+1} \cong I_g^t/I_g^{t+1} \otimes_{S_g} S \cong I_g^t/I_g^{t+1} \otimes_{S_g/I_g} S/I$. Since I_g, I_L, I are all Artinian, $S_g/I_g \cong S_L/I_L \cong S/I$. This proves (4). (4) implies (5) by the definition of strongly Lech-independence and expansion property. For (6), it suffices to check that $I^t \cap S_L = I_L^t$ and $I_L^t \cap S_g = I_g^t$. The first equality is true because i_2 is faithfully flat. For the second equality, pick $f \in I_L^t \cap S_g$, then there exist $f' \in S_g \backslash \mathfrak{n}_g$ such that $ff' \in I_g^t$. Then f' has a nonzero constant term f'_0. Let the term of f with lowest degree s be f_s. Then the term of ff' with lowest degree is $f'_0 f_s$ which has degree s, so $s \geq t$, so $f \in I_g^t$, which completes the proof. For (7), we have $e(I_g) = e(I_L) = e(I)$ by (4) and the second part of (7) can be proved by taking $I_g = \mathfrak{n}_g$. □

Theorem 4.7. *Let $(S, \mathfrak{n})$ be the localization of a standard graded ring $(S_g, \mathfrak{n}_g)$ over a field k. Let I be an $\mathfrak{n}$-primary Lech-independent S-ideal which is extended from a homogeneous ideal I_g with homogeneous minimal generators $x_1, ..., x_r$ such that x_i is homogeneous in S_g of degree t_i and $t_1 \leq t_2 \leq ... \leq t_r$. Assume that there is a standard set Γ such that $x_1, ..., x_r$ is Γ-expandable. Then $e(S) \geq e(\Gamma) t_1 ... t_{r-d}$.*

In particular, if there is a flat local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that $I = \mathfrak{m}S$ then $e(S) \geq e(R)t_1...t_{r-d}$.

Proof. By Proposition 4.6 we may always complete S to assume that S is the completion of S_g with respect to $\mathfrak{n}_g$, and everything in the assumptions and conclusions are not affected. Moreover in S we have $\mathrm{ord}(x_i) = t_i$. Since I is homogeneous, we may choose a k-basis f_i of S/I such that each f_i is homogeneous in S_g; here we view S_g as a subring of S. Also the homogeneous minimal generators $x_1, ..., x_r$ are in S_g. Now let $\sum c_{i,u} f_i u(x)$ be a sum satisfying $c_{i,u} \in k, u \in \Gamma$, where c_i's are not all 0, and $\mathrm{ord}(f_i) + \sum_j \mathrm{ord}(x_j) \deg_{T_j}(u) < t$ for any $c_{i,u} \neq 0$. Then the sum is nonzero by unique expansion property. Also, each term is in S_g and we can view the sum as an element in S_g; now each term has nonzero components only in degree smaller than t. So the sum has nonzero components in degree smaller than t, and in particular, it does not lie in $\mathfrak{n}_g^t$, so it does not lie in $\mathfrak{n}^t$ because $\mathfrak{n}^t \cap S_g = \mathfrak{n}_g^t$. So $\{f_i u(x), \mathrm{ord}(f_i) + \sum_j \mathrm{ord}(x_j) \deg_{T_j}(u) < t\}$ is k-linearly independent modulo $\mathfrak{n}^t$. Since this is true for any t, Theorem 4.5(4) implies that $e(\mathfrak{n}) \geq e(\Gamma) l(S/I)/t_r t_{r-1}...t_{r-d+1}$. Now assume there is a flat local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that $I = \mathfrak{m}S$. Note that $\mathfrak{m}$ is strongly Lech-independent, so I is strongly Lech-independent, so it is Lech-independent, so by Hanes' result in [4], $l(S/I) \geq t_1 t_2 ... t_r$. So $e(\mathfrak{n}) \geq e(\Gamma) t_1 t_2 ... t_r / t_r t_{r-1} ... t_{r-d+1} = e(\Gamma) t_1 t_2 ... t_{r-d}$. Also $\mathfrak{m}$ is Γ'-expandable for some Γ', so I is also Γ'-expandable. This implies $e(R) = e(\Gamma') = e(\Gamma)$ by Proposition 3.11. So $e(\mathfrak{n}) \geq e(R) t_1 t_2 ... t_{r-d}$. □

Remark 4.8. Theorem 4.7 is a generalization of some of Hane's results, for example, Corollary 3.2 of [4]. We make no assumptions on the minimal reduction of $\mathfrak{m}$ or $\mathfrak{m}S$. For example, consider $R = k[[x, y^2]]/xy^2 \to S = k[[x, y]]/xy^2$. Then neither x or y^2 can be a minimal reduction of $\mathfrak{m}$. The minimal reduction consists of one element which is a linear combination of x and y^2 which is not homogeneous. So we cannot use Hane's result, but we can apply Theorem 4.7 to prove $e(R) \leq e(S)$.

We can strengthen the first inequality in Theorem 4.5 (4) using the asymptotic Samuel function.

Definition 4.9. The asymptotic Samuel function is $\bar{v} : S \to \mathbb{R} \cup \{\infty\}$ such that $\bar{v}(x) = \lim_{n \to \infty} \mathrm{ord}(x^n)/n$.

Proposition 4.10. *Let S be a local ring.*
(1) $\bar{v}$ is well-defined, that is, the limit exists for any $x \in S$.
(2) $\bar{v}$ has values in $\mathbb{Q} \cup \{\infty\}$.
(3) $\bar{v}(x) \geq \mathrm{ord}(x)$.

Proof. For (1) (2) see Chapter 6 and 10 of [7]. (3) is true as $\mathrm{ord}(x^n) \geq n \cdot \mathrm{ord}(x)$. □

Proposition 4.11. *Let I be an ideal in S which is $\mathfrak{n}$-primary. Assume $I = (x_1, ..., x_r)$ and the sequence $x_1, ..., x_r$ is Γ-expandable with* $\dim(\Gamma) = d > 0$. *Denote $\bar{v}(x_i) = s_i$ and assume that $s_1 \leq s_2 \leq ... \leq s_r$. Then $e(\mathfrak{n}) \leq e(\Gamma) l(S/I)/s_1 ... s_{d-1} s_d$ and $s_d < \infty$.*

Proof. Choose any positive rational number $q_i < s_i$. Choose a positive integer C such that Cq_i is an integer for any i. Take $f_1, f_2, \ldots, f_l$ such that their images form a k-basis of S/I. By definition of $s_i = \bar{v}(x_i)$, there exist a constant D_i in $\mathbb{Z}$ such that $\mathrm{ord}(x_i^n) \geq nq_i + D_i$. So if $u = T_1^{a_1} T_2^{a_2} ... T_r^{a_r}$, $\mathrm{ord}(u(x)) \geq q_1 a_1 + q_2 a_2 + ... + q_r a_r + D$, where $D = D_1 + D_2 + ... + D_r$. Let $Z = z^{1/C}$ and view $\mathbb{R}[[z]]$ as a subring of $\mathbb{R}[[Z]]$.

Let $b(z)=\sum_{t\geq 0,t\in\mathbb{Q}} b_t z^t$ where b_t is the number of $f_i u(x)$ such that $\sum_j q_j \deg_{T_j}(u)=t$ then similar as Theorem 4.5 (1) we can prove $b(z)=l(S/I)HS_\Gamma(z^{q_1},z^{q_2},...,z^{q_r})$. The exponents of terms in $b(t)$ is in $1/C\mathbb{Z}$, so we can view

$$b(z)=b(Z^C)=l(S/I)HS_\Gamma(Z^{Cq_1},Z^{Cq_2},...,Z^{Cq_r})$$

as an element of $\mathbb{R}[[Z]]$.

Assume $t\in 1/C\mathbb{Z}$. Let $A_{3,t}$ be the set $\{f_iu(x)|1\leq i\leq l, u\in\Gamma, \sum_j q_j\deg_{T_j}(u)<t\}$. Note that $A_{3,t}\subset A_{3,t+1/C}$ and

$$A_{3,t+1/C}\backslash A_{3,t}=\{f_iu(x)|1\leq i\leq l, u\in\Gamma, \sum_j q_j\deg_{T_j}(u)=t\}$$

because $C\sum_j q_j\deg_{T_j}(u)$ is always an integer. This implies $|A_{3,t+1/C}|-|A_{3,t}|=b_t$. Consider the series

$$b'(z)=\sum_{t\geq 0,t\in 1/C\mathbb{Z}}|A_{3,t}|z^t=\sum_{t\geq 0,t\in\mathbb{Z}}|A_{3,t}|Z^{Ct}.$$

Then $b'(z)=b(z)(1-z^{1/C})$ or equivalently, $b'(Z^C)=b(Z^C)(1-Z)$. The Hilbert series of S is

$$HS_S(z)=\sum_i \dim_k(\mathfrak{n}^i/\mathfrak{n}^{i+1})z^i=HS_S(Z^C)=\sum_i\dim_k(\mathfrak{n}^i/\mathfrak{n}^{i+1})Z^{Ci}.$$

Let $a(Z)=\sum_i a_iZ^i=HS_S(Z^C)(1-Z)$. Then $a_i=\dim_k(S/\mathfrak{n}^{\lfloor i/C\rfloor+1})$ where $\lfloor\cdot\rfloor$ is the floor function.

Suppose $t\in\mathbb{Z}$. Since $\operatorname{ord}(f_iu(x))\geq q_j\deg_{T_j}(u)+D$, $\sum_j q_j\deg_{T_j}(u)\geq t$ implies $f_iu(x)\in\mathfrak{n}^{t+D}$, so $S/\mathfrak{n}^{t+D}$ can be spanned by $A_{3,t}$. This means that $\dim_k(S/\mathfrak{n}^{t+D})\leq|A_{3,t}|$. So if t is an integer

$$a_{Ct+CD-C}=\dim_k(S/\mathfrak{n}^{t+D})\leq|A_{3,t}|.$$

As $|A_{3,t}|$ is increasing in terms of t and a_i only depends on $\lfloor i/C\rfloor$,

$$a_{Ct+CD-C}=a_{C\lfloor t\rfloor+CD-C}\leq|A_{3,\lfloor t\rfloor}|\leq|A_{3,t}|$$

for any $t\in 1/C\mathbb{Z}$, or equivalently, $a_{t+CD-C}\leq|A_{3,t/C}|$ for any $t\in\mathbb{Z}$. This means that

$$\sum_{t\geq 0,t\in\mathbb{Z}}a_{t+CD-C}Z^t\leq\sum_{t\geq 0,t\in\mathbb{Z}}|A_{3,t/C}|Z^t.$$

So

$$Z^{C-CD}HS_S(Z^C)/(1-Z)+P(Z)\leq b(Z^C)/(1-Z) \tag{4.5}$$

where $P(z)$ is the term of $Z^{C-CD}HS_S(Z^C)/(1-Z)$ with negative exponents; in particular $P(z)$ is a Laurent polynomial in z. On the left side of (4.5), $HS_S(z)$ has a single pole at $z=1$ of order d; so $HS_S(Z^C)$ has a pole at $z=\xi$ of order d for every C-th root of unity where we view Z as the variable. This implies that $Z^{C-CD}HS_S(Z^C)/(1-Z)+P(Z)$ has a pole at $Z=1$ of order $d+1$ and a pole at $Z=\xi$ of order d for every C-th root of unity $\xi\neq 1$. This means that $Z^{C-CD}HS_S(Z^C)/(1-Z)+P(Z)$ satisfies (P1), (P2_{d+1}), and (P3_{d+1}). On the right side of (4.5), we have

$$b(Z^C)/(1-Z)=l(S/I)HS_\Gamma(Z^{Cq_1},Z^{Cq_2},...,Z^{Cq_r})/(1-Z)$$

and by the same proof in Theorem 4.5 (3) we know $b(Z^C)/(1-Z)$ also satisfies (P1), (P2$_{d+1}$), and (P3$_{d+1}$). Now apply Lemma 4.4, we get

$$\lim_{Z\to 1}(Z^{C-CD}HS_S(Z^C)/(1-Z)+P(Z))(1-Z)^{d+1} \le \lim_{Z\to 1} b(Z^C)(1-Z)^d. \tag{4.6}$$

The left side of (4.6) is equal to

$$\lim_{z\to 1}(z^{1-D}HS_S(z)/(1-z^{1/C})+P(z^{1/C}))(1-z^{1/C})^{d+1}$$
$$= \lim_{z\to 1} HS_S(z)(1-z^{1/C})^d = 1/C^d \cdot \lim_{z\to 1} HS_S(z)/(1-z)^d = 1/C^d e(\mathfrak{n}).$$

The right side of (4.6) is equal to

$$l(S/I)\sum_{i,|S_i|=d}\frac{1}{\Pi_{T_j\in S_i}Cq_j} = l(S/I)/C^d \cdot \sum_{i,|S_i|=d}\frac{1}{\Pi_{T_j\in S_i}q_j}$$

which is smaller than $1/C^d\cdot e(\Gamma)l(S/I)/q_1...q_{d-1}q_d$ by a similar proof in Theorem 4.5 (3) and (4). So multiplying (4.6) by C^d we get $e(\mathfrak{n}) \le e(\Gamma)l(S/I)/q_1...q_{d-1}q_d$. Let q_i goes to s_i we get $e(\mathfrak{n}) \le e(\Gamma)l(S/I)/s_1...s_{d-1}s_d$. But $e(\mathfrak{n}) > 0$, so $s_d < \infty$. □

By proposition 4.10 (3) $s_i = \bar{v}(x_i) \ge t_i = ord(x_i)$, so Proposition 4.11 is stronger than Theorem 4.5 (4).

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References

[1] J. Herzog, B. Ulrich, and J. Backelin: Linear maximal Cohen-Macaulay modules over strict complete intersections, J. Pure Appl. Algebra 71 (1991), no. 2-3, 187-202. 1117634

[2] D. Eisenbud, Commutative algebra with a view towards algebraic geometry. Volume 150 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[3] D. Hanes: Special conditions on maximal Cohen-Macaulay modules, and applications to the theory of multiplicities, Thesis, University of Michigan (1999).

[4] D. Hanes: Length approximations for independently generated ideals, J. Algebra 237 (2001), no. 2, 708-718. 1816712

[5] B. Herzog: Lech-Hironaka inequalities for flat couples of local rings, Manuscripta Math. 68 (1990), no. 4, 351-371. 1068260

[6] B. Herzog: Kodaira-Spencer maps in local algebra, Lecture Notes in Mathematics, vol. 1597, Springer-Verlag, Berlin, 1994. 1329544

[7] I. Swanson and C. Huneke, Integral Closure of Ideals, Rings, and Modules, London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006

[8] E. Kunz: On Noetherian rings of characteristic p, Amer. J. Math. 98 (1976), no. 4, 999-1013. 0432625

[9] C. Lech: Note on multiplicities of ideals, Ark. Mat. 4 (1960), 63-86 (1960). 0140536

[10] C. Lech: Inequalities related to certain couples of local rings, Acta Math. 112 (1964), 69-89. 0161876

[11] L. Ma: The Frobenius endomorphism and multiplicities, Thesis, University of Michigan (2014).

[12] L. Ma: Lechs conjecture in dimension three, Adv. Math. 322 (2017), 940-970. 3720812

[13] L. Ma: Lim Ulrich sequence: proof of Lech's conjecture for graded base rings, arXiv:2005.02338.

[14] K.E.Smith, Characteristic p Techniques in Commutative Algebra and Algebraic Geometry, Lecture notes available at `http://www.math.lsa.umich.edu/~kesmith/732CourseNotes.pdf`

[15] W.V.Vasconcelos, Computational Methods in Commutative Algebra and Algebraic Geometry, Springer-Verlag, Berlin, 1998.